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Abstract: *Avoiding complex geometric and analytic constructions, this paper considers techniques for visualizing the location of complex roots of quadratic, cubic, and quartic real polynomial functions. This provides teachers and students of mathematics with a better understanding of the nature of these functions and their respective real and complex roots.*

Keywords: Quadratic, Cubic, Quartic, Polynomials, Complex Roots

We are well aware that the real roots of a polynomial function are recognized as the *x*-intercepts of the associated graph and that some quadratics have complex, non-real roots, always appearing in conjugate pairs. Curiosity naturally arises regarding the location of these complex roots. Herein, based singularly on the graphs (no equations) of quadratic, cubic, and quartic polynomial functions, we investigate means of visualizing exact or impressively precise approximations of the locations of complex roots. To accomplish this, we first consider some preliminary mathematical notions.

Mathematical Preliminaries

In this investigation, numerous mathematical ideas and theorems are denoted with #; for these, ancillary online materials provide proofs and additional information.

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Throughout this paper, the term *polynomial* denotes real monic polynomials, or polynomials with real coefficients in which the leading coefficient is 1. (Non-monic real polynomials can be divided by the leading coefficient to make them monic, without affecting the roots.) These polynomials are graphed on the Cartesian plane ($R \times R$). Since non-real complex points belong to the complex plane, we co-label the y-axis with both real and imaginary values such that the complex root $a+b\imath$ is located as (a, b) on a *Superimposed Plane*. This plane retains the real-valued x-axis of the usual Cartesian plane (allowing for real zeros of polynomials to be treated in the usual manner) and has a y-axis that can be used in respect to locating the complex roots.

Herein, we investigate means of analytically and geometrically constructing and locating complex roots on quadratic, cubic, and quartic polynomial functions. In some cases, these locations are exact and in other cases they are impressively precise approximations. As will be seen through some novel techniques and only minimal calculations, students can quite readily visualize the location of these complex roots and can come to appreciate the beauty of polynomial functions in a unique way, often invisible to those who only consider the real Cartesian plane.

Constructing the Complex Roots on Quadratic Functions

When quadratic polynomial functions have a conjugate pair of complex, non-real roots, a simple construction reveals their locations (Norton & Lotto, 1984).

Technique: Begin with the graph of a quadratic function $f(x)$ with complex roots. (See Figure 1.) Through the vertex, construct a line parallel to the x-

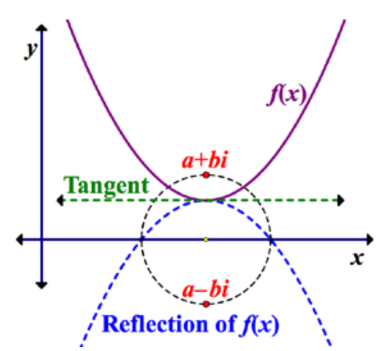

Figure 1

axis. Reflect the parabola across this line. This reflection will intersect the *x*-axis. Construct a circle with diameter points at these *x*-intercepts and rotate the intercepts about the center of the circle by 90° (i.e., multiply by *i*). These two vertically stacked points (*a*, *bi*) and (*a*, *-bi*) are the complex roots, *a*+*bi* and *a*–*bi*, of *f*(*x*) in the Superimposed Plane. Thus, we can recognize that $f(x)=(x-(a+bi))(x-(a-bi))=(x-a)^2+b^2$. Notably, since (*a*, b^2) is the parabola's vertex, the vertex and the complex roots will be vertically collinear. Further, since $f(a)=(a-a)^2+b^2=b^2$, then we can readily calculate the height of the complex root with $b=\pm\sqrt{f(a)}=\pm\sqrt{\text{ordinate of vertex of } f(x)}$. (To investigate this through a dynamic graph use:

http://appstate.edu/~bossemj/VisualizingRoots/MT/Quadratics/.)

Constructing Complex Roots on Cubic Functions

When a cubic polynomial function has one real root and two complex roots, a simple method allows us to visually locate the complex roots. We will begin with a cubic, *g*(*x*), with a real root at *r* and two complex roots, where the real part of the complex roots is not equal to *r*.

Technique: Construct a tangent from the real root (*r*, 0) to *g*(*x*) and denote the point of tangency as (*j*, *k*) (Ward, 1937). (See Figure 2.) This tangent is unique. Determine the slope (*m*) of this tangent. The imaginary roots will be at $\left(j, \pm i\sqrt{m}\right)$ on the Superimposed Plane

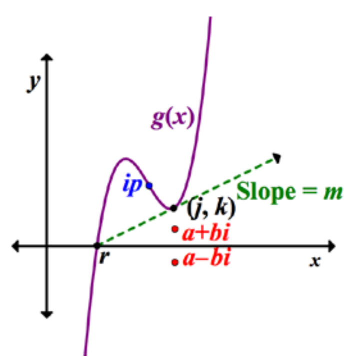

Figure 2

and vertically collinear with the point of tangency. Since $g(a) = b^2(x - r)$, we can see that $b = \sqrt{m} = \sqrt{g(a)/(a-r)}$. (Notably, although the point of tangency is in the neighborhood of the extremum, these points are generally not the same.) The inflection point (denoted *ip* on Figure 2) occurs at $x_i = \frac{1}{3}(r + a)$, or the inflection point is 2/3 of the way from the real root to the *x*-value of the complex root. When *r* approaches *j*, the extremum approaches the inflection point. (To investigate this through a dynamic graph use: http://appstate.edu/~bossemj/VisualizingRoots/MT/Cubics/.)

While this technique works analytically in all cases in which *r*≠*j* and all roots are not vertically collinear, geometrically visualizing the tangent when the real root is close to the real value of the complex roots may not be possible. When *r*=*j*, the curve has no extrema, *r* is an inflection point, and the formula holds in the limit with $b = \sqrt{g'(r)} = \sqrt{g'(j)}$.

Constructing Complex Roots for Quartic Functions

Quartic real polynomials with complex roots come in two forms: two real roots (whether distinct or equal) and two complex roots (and these complex roots may or may not be vertically collinear with one or both real roots); or all four non-real complex roots (that may or may not be all vertically collinear). A common characteristic that can be recognized in the graphs of most quartics investigated in high school and college is that they possess three extrema (i.e., two relative minima and one relative maximum). However, beyond this common type of quartic, the visualization technique provided herein successfully addresses a far more broad classification of quartics: any quartic on which a bitangent (a line that is tangent to the graph at two points) can be constructed. (Bitangents are depicted in some subsequent figures.)

However, before we begin, some ideas need to be briefly addressed. First, not all quartics have a bitangent, and even when they do, they are not always obviously visible. Second, when a bitangent exists, it is unique. In order to investigate this more thoroughly, we propose the following definition and theorems:

Definition: A quartic polynomial with no third degree term is called reduced. Any quartic polynomial, $q(x)=\sum_{i=0}^{4} a_i x^i$, may be reduced with the substitution

$$\hat{q}(x)=q\left(x-\frac{a_3}{4a_4}\right)=a_4x^4+b_2x^2+b_1x+b_0.$$

Theorem 1. Let p be the reduced quartic polynomial $p(x)=x^4+a_2x^2+a_1x+a_0$. If $a_2<0$, then the line $B(x)=a_1x+\left[a_0-\frac{a_2^2}{4}\right]$ is the unique bitangent to p with points of tangency at $x=\pm\sqrt{-a_2/2}$.

Theorem 2. For real monic quartic polynomials (including non-reduced), if the following conditions are met, the quartic will possess a bitangent. For

❖ $\text{NoReals}(x)=\left[(x-a)^2+b^2\right]\left[(x-c)^2+d^2\right]$,

$$b^2+d^2-\frac{(a-c)^2}{2}<0.$$

❖ $\text{TwoReals}(x)=\left[(x-a)(x-r)\right]\left[(x-c)^2+d^2\right]$,

$$d^2-\frac{(a-r)^2+2(a-c^2)+2(c-r)^2}{8}<0.$$

❖ $\text{AllReals}(x) = \left[(x-a)(x-r_1)(x-c)(x-r_2)\right]$,

$$(a-c)^2 + (a-r_1)^2 + (a-r_2)^2 + (c-r_1)^2 + (c-r_2)^2 + (r_1-r_2)^2 > 0.$$

The *x*-coordinates of the bitangent are

$$x_b = \frac{1}{4}\left[(a+r_1+c+r_2) \pm \sqrt{(a-c)^2 + (a-r_1)^2 + (a-r_2)^2 + (c-r_1)^2 + (c-r_2)^2 + (r_1-r_2)^2}\right].$$

Theorem 3. The only case of a monic, real, quartic polynomial with all real roots not having a bitangent is when the four roots are the same; i.e., a real root of multiplicity 4: $\text{AllReals}(x) = (x-a)^4$.

To remedy difficulty locating the bitangent, construct one of the following lines (See Figure 3.) and slide it down (retaining the same slope) until it becomes a bitangent: (***T***) If the quartic has two real roots (whether or not distinct), r_1 and r_2, draw the line tangent to the quartic at $x = \frac{1}{2}(r_1 + r_2)$. (***I***) Draw a line through the graph's points of inflection. Lines ***T*** and ***I*** are parallel to the bitangent (***B***).

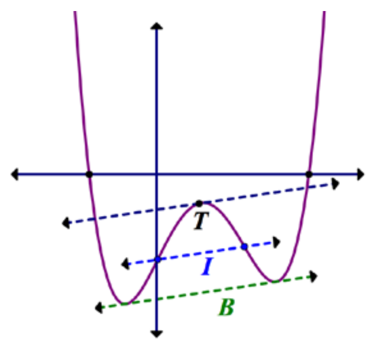

Figure 3

With this background in place, this investigation is restricted to cases in which the bitangent is obviously visible. These are considered in two broad cases: when the function has two real roots (double or distinct) or no real roots (Yanosik, 1943). (To investigate all cases of quartics through a dynamic graph use: http://appstate.edu/~bossemj/VisualizingRoots/MT/Quartics/.)

Quartics with Two Real Roots

We begin with a quartic function, $h(x)$, with real roots at r_1 and r_2 and complex roots $a \pm bi$, where $r_1 \neq a$ and $r_2 \neq a$. In all these cases, since $h(a) = b^2(a - r_1)(a - r_2)$, $b = \pm\sqrt{h(a)/((a - r_1)(a - r_2))}$; thus, the height of the complex root is readily calculated.

When the real roots are distinct: Determine the x-values of the two inflection points. (We denote ip_1 and ip_2 as both the inflection points and their respective x-values.) (See Figure 4.) Then, a remarkably simple calculation produces the real part of the complex roots: $a = ip_1 + ip_2 - \frac{1}{2}(r_1 + r_2)$. As seen in Figure 5, with bitangent points at (f, g) and (j, k), this equation can be written as $a = \frac{2}{3}(f + j) - \frac{1}{2}(r_1 + r_2)$. While we now have the locations of the complex roots, some beautiful additional findings come to light.

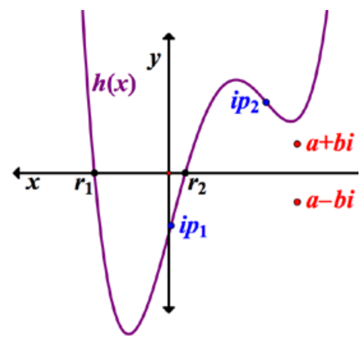

Figure 4

First, as seen in Figure 5, when one point of inflection is between the real roots (ip_1) and another is outside the real roots (ip_2) the complex roots are vertically in the neighborhood of the uppermost point of bitangency; j approximates the value of a. Or, $a = j \pm \Delta$.

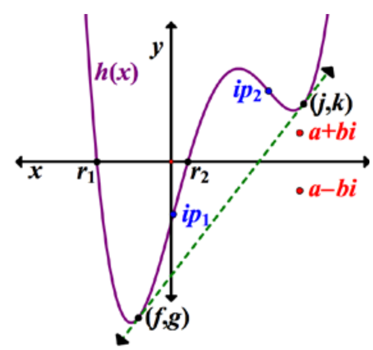

Figure 5

Second, if a relative max at $x=c$ exists between r_1 and r_2, then $a+bi$ resides in the complex rectangular strip bounded on three sides by $x=c-\Delta$,

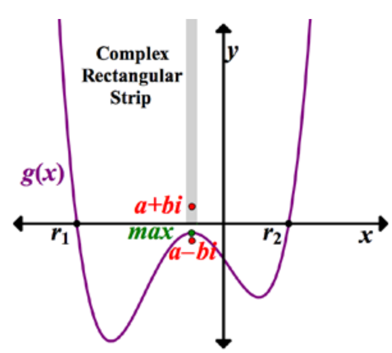

Figure 6

$x=c+\Delta$, and $y=0$. (See Figure 6.) More generally, a is in the neighborhood of the relative maximum between the real roots.

When the real roots are equal: Let $r_1=r_2=r$ and $r\neq a$. We can employ a tangent similar to the cubic: Construct a tangent from $(r, 0)$ to $h(x)$. Then the complex roots are vertically in the neighborhood of the point of tangency, (j,k); j approximates the value of a. Or, $a=j\pm\Delta$.

Quartic With No Real Roots

Again, we begin with quartics with obviously visible bitangents. Without loss of generality, we begin with a quartic such that the left relative minimum is less than or equal to the right relative minimum. (See Figure 7.) Construct the bitangent with the points of tangency of (f, g) and (j, k) such that $f<j$; this will lead to $g\leq k$. In this case, $a+bi$, resides in the complex rectangular strip bounded on three sides by $x=f-\Delta$ and $x=f$, and $y=0$ and $c+di$ resides in the complex rectangular strip bounded by $x=j$ and $x=j+\Delta$. If $g<k$, the maximum height of the complex rectangular strip for $a+bi$ is the minimum height of the complex rectangular strip for $c+di$. Unfortunately, determining these upper and lower bounds for these complex rectangular strips require analytic constructions that are significantly beyond what can be visualized. If $g=k$, then the lower bounds for both strips is $y=0$. Note that these relationships change accordingly when the left relative minimum is greater than the right relative minimum.

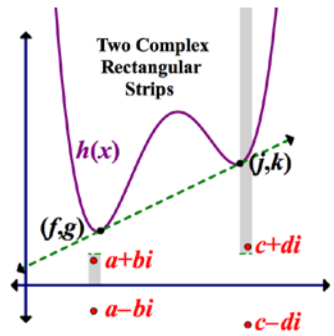

Figure 7

How Large is Δ?

At this point, some may be prone to ask how large or small a value Δ may be. Recall that in the cases of the quadratic and the cubic, along with the symmetric quartic with no real roots, the techniques provided exact locations for the real part of the complex roots and a rectangle was unnecessary. This can be restated as: in these cases, the rectangles had a width of zero.

It is easy to state that in all cases Δ is small – indeed, surprisingly small and often approaching zero. Unfortunately, the mathematics to justify this claim instantaneously transcends many mathematically sophisticated audiences and becomes more an obstacle than valuable information. We would need to discuss theorems from: Lagrange (1789); Cauchy (1829); Rouché (1862); the Gauss-Lucas Theorem (1874); Mohammad (c1980); Kojima (1914); Fujiwara (1916); and many others. It suffices to say that the search for these roots is far from novel and mathematicians have long recognized this search as valuable.

Therefore, Δ affords us to be able to visualize an amazingly precise approximate location for these complex roots based singularly on the graph of the respective polynomial with minimalist constructions. This places deep, rich, meaningful, and intriguing mathematics within the grasp of high school and college students.

Summary and Conclusion

The visualizing techniques provided above allow students to investigate the location of complex roots of quadratic, cubic, and quartic monic polynomial functions and shed light on concepts previously left mostly cloaked in curricular silence. The common notion of tangents and bitangents elegantly tie the techniques together. It is hoped that this brief investigation accomplishes some or all of the following:

1. Mathematics teachers can now go beyond stating that polynomials of degree *n* have *n* complex roots (some being real), they can demonstrate where these are in respect to quadratics, cubics, and quartics.

2. Students who have previously been curious about the locations of complex roots may gain further insight into this topic and may even wish to investigate this further through: working through some of the theorems provided through the ancillary documents previously mentioned; considering other construction techniques and why they work; challenging themselves to consider these notions in respect to higher degree polynomials; and hybridizing other graphing modalities that may be even more informative for this and other mathematical investigations.

Students who have access to a 3D graphing utility may gain further insights by plotting the magnitude of the complex polynomial $|h(x+iy)|$. Figure 12 shows the 3D surface hi-lighting the roots; the roots are on the complex plane. Notably, the real roots are on the *x*-axis (or the real axis) and the complex roots are clearly visible at $-1 \pm i$ on the complex plane. Figure 13 shows the surface's contours encircling the real and complex roots from a viewpoint on the *z*-axis.

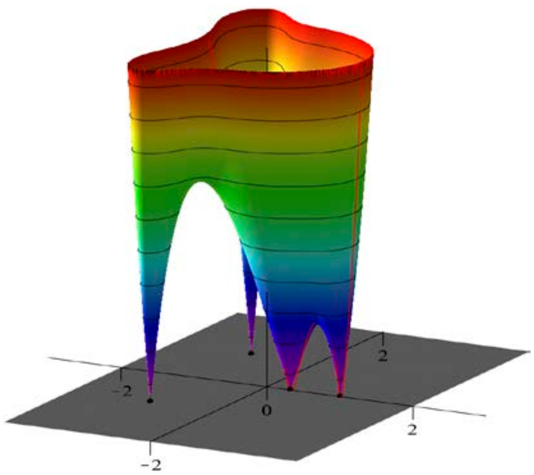

Figure 12

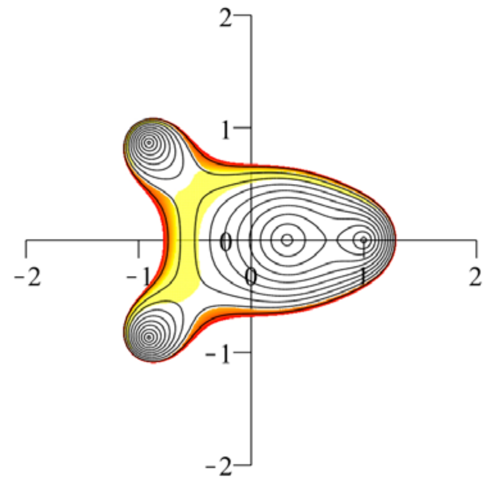

Figure 13

The reader may now wonder if this story is complete. What about polynomials of higher degree? Since Abel (1823) proved that that there is no general algebraic solution to polynomials equations of degree five or higher with arbitrary coefficients, our story is almost complete (Rosen, 1995). We cannot visualize the location of complex roots of polynomials of degree higher than the quartic apart from polynomials with the special form $f(x)=(x-r)^n\left((x-a)^2+b^2\right)$, where r is the real root and $a \pm bi$ the complex roots. In this special case, the tangent technique employed for the cubic works well (Ellis, Bauldry, Bossé, & Otey, 2016).

Additional extensions to the ideas in this paper can be found in other papers recently written by the authors (Bauldry & Bossé, 2018; Bauldry, Bossé, & Otey, 2017; Bossé, & Bauldry, & Otey, 2018).

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